SET THEORY HOMEWORK 1

Due Monday, September 16.

Recall that α is an ordinal if $(\alpha, \in)$ is a well order and α is transitive (i.e. $\beta \in \alpha \to \beta \subset \alpha$).

Problem 1. *Let y be a set of ordinals.*

1. *If y is nonempty, show that the $\in$-minimal element in y is unique.*
2. *Show that $\bigcup y$ is an ordinal.*

Problem 2. *Suppose that $(x, <)$ is a well order. Show that there is a unique ordinal α, such that $o.t.(x, <) = \alpha$, i.e. there is an order preserving bijection $f : (x, <) \to (\alpha, \in)$.*

Hint: for the existence part, for each $y \in x$, define $pred(y) = \{z \in x \mid z < y\}$ and set $pred(x) = x$. Prove the stronger statement that for every $y \in x \cup \{x\}$, there is a order preserving bijection between $pred(y)$ and some ordinal.

Recall that for ordinals α, β, we defined $\alpha + \beta = o.t.(x, <)$, where $x = \{\langle 0, \xi\rangle \mid \xi < \alpha\} \cup \{\langle 1, \xi\rangle \mid \xi < \beta\}$ and $\langle i, \xi\rangle < \langle j, \xi'\rangle$ if $i < j$ or $i = j$ and $\xi < \xi'$. We also defined $\alpha \cdot \beta = o.t.(\beta \times \alpha, <_{lex})$ where $\langle \delta, \xi\rangle <_{lex} \langle \delta', \xi'\rangle$ if $\delta < \delta'$ or $\delta < \delta'$ and $\xi < \xi'$ (i.e. this is β many copies of α).

For the following problems assume that $\alpha, \beta, \gamma, \delta, \xi$ are ordinals.

Problem 3. *Show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$\alpha \leq \beta \to (\exists! \delta)(\alpha + \delta = \beta).$$

Problem 4. *Show that if $\gamma > 0$, then $\alpha < \beta$ implies that $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$(\alpha \leq \beta \wedge \alpha > 0) \to (\exists! \delta, \xi)(\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta).$$